Feb 23, 2017

Thursday, February 23, 2017 6:06 PM

Baire Category Theorem

Every complete metric space is of 2^{nd} category.

Consequence

$\mathbb{R}$, and thus $\mathbb{R}\setminus\mathbb{Q}$, are of 2^{nd} category.

Idea. Let us start with some intuition.

Recall that $(\bar{N})^\circ = \phi \Leftrightarrow X\setminus\bar{N}$ is dense

Even further, $\forall\ U \in \mathcal{J}$, $U\setminus\bar{N}$ is dense in U.

In the case that $X = \bigcup_{k=1}^{\infty} N_k$ with $(\bar{N}_k)^\circ = \phi$,

$X\setminus\bar{N}_1$ is dense, and open.

$\therefore$ $(X\setminus\bar{N}_1)\setminus\bar{N}_2$ is dense in $X\setminus\bar{N}_1$, and so on.

Inductively, $\bigcap_{k=1}^{\infty}(X\setminus\bar{N}_k) = \phi$ but "dense" at any finite step.

We hope to create a contradiction if X is a complete metric space.

The only reasonable tools are

Cauchy Sequence

Contraction Mapping

Cantor Intersection (Nested closed sets)

No matter which one we use, the aim is to create an element in $\bigcap_{k=1}^{\infty}(X\setminus N_k)$.

02-23-p2

Thursday, February 23, 2017 6:07 PM

Proof. Assume $X = \bigcup_{k=1}^{\infty} N_k$, each $(\overline{N_k})^\circ = \phi$

Since $X \setminus \overline{N_1}$ is dense, pick $x_1 \in X \setminus \overline{N_1}$

Also, $X \setminus \overline{N_1}$ is open, $x_1 \in B(x_1, 2r_1) \subset X \setminus \overline{N_1}$

Let $F_1 = \{x \in X : d(x, x_1) \leq r_1\}$

Consider the open set $B(x_1, r_1) \subseteq F_1$,

$B(x_1, r_1) \setminus \overline{N_2} \neq \phi$, pick $x_2 \in B(x_1, r_1) \setminus \overline{N_2}$

and $x_2 \in B(x_2, 2r_2) \subset B(x_1, r_1) \setminus \overline{N_2}$

and $F_2 = \{x \in X : d(x, x_2) \leq r_2\} \subset B(x_1, r_1) \subset F_1$

This process is inductively repeated to have

$x_{k+1} \in B(x_{k+1}, 2r_{k+1}) \subset B(x_k, r_k) \setminus \overline{N_{k+1}}$

$F_{k+1} = \{x \in X : d(x, x_{k+1}) \leq r_{k+1}\} \subset B(x_k, r_k) \subset F_k$

By Cantor Intersection Theorem,

$\phi \neq \bigcap_{k=1}^{\infty} F_k \subset \bigcap_{k=1}^{\infty} (X \setminus \overline{N_k})$

That is a contradiction.

02-23-p3

Thursday, February 23, 2017 6:07 PM

Finite Product Given $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$.

The product topology for $X \times Y$ is generated by $\mathcal{S} = \{U \times Y : U \in \mathcal{T}_X\} \cup \{X \times V : V \in \mathcal{T}_Y\}$

and it is denoted by $\mathcal{T}_{X \times Y}$

After taking finite intersections on $\mathcal{S}$, we have a base $\mathcal{B} = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$

Example. $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$

One may imagine that at every point $(x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$, a copy of $\mathbb{R}$ is placed.

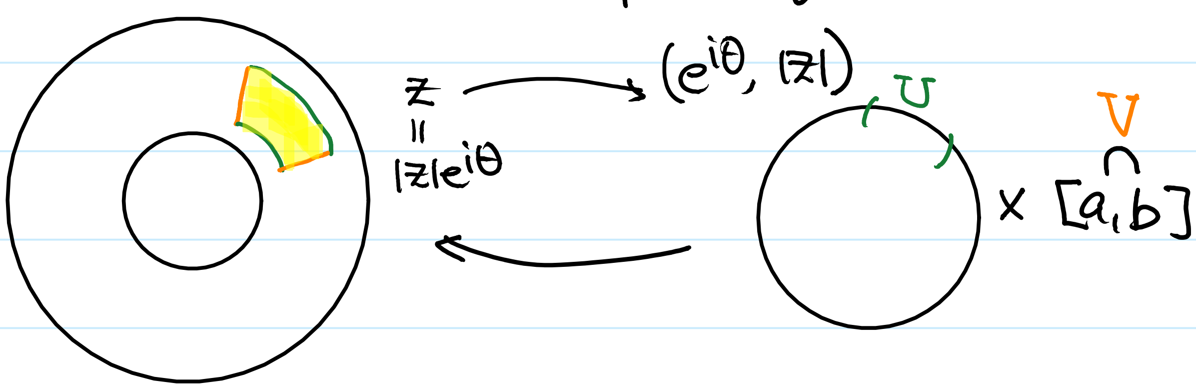
The "basic" open set of $\mathbb{R}^n$ is $U \times (a,b)$ where $(x_1, \cdots, x_{n-1}) \in U \subset \mathbb{R}^{n-1}$ and $(a,b) \subset \mathbb{R}$.

Example. Annulus and Cylinder

Let $A = \{z \in \mathbb{C} : a \leq |z| \leq b\} \subset \mathbb{C} = \mathbb{R}^2$

and $S^1 = \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C} = \mathbb{R}^2$

Both A and S^1 are subspace of $\mathbb{C} = \mathbb{R}^2$

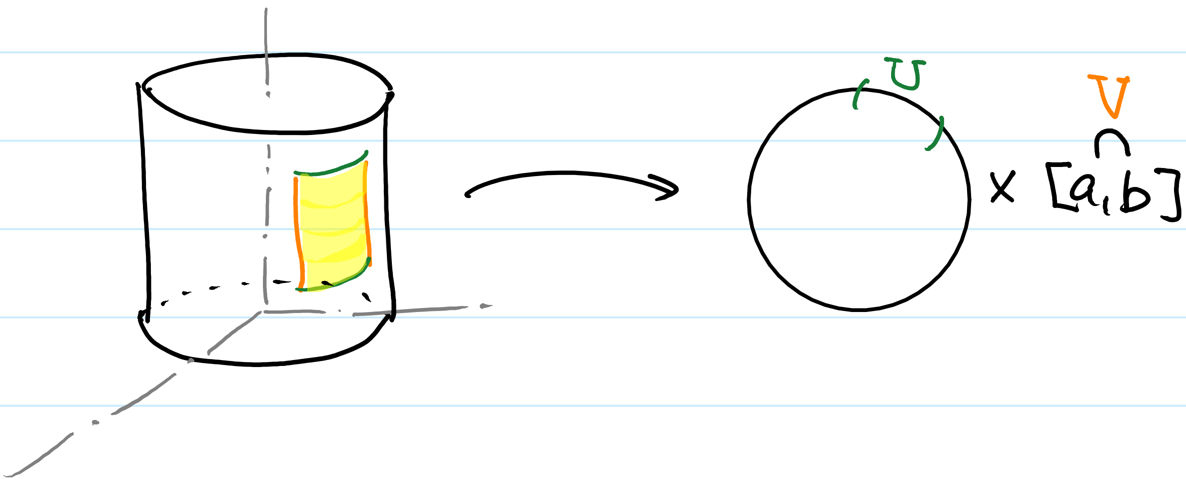


The cylinder, C, can be describe as a subspace in $\mathbb{R}^3$.

$$C = \{(u,v,w) \in \mathbb{R}^3 : u^2+v^2=1,\ w \in [a,b]\}$$

$$C \xrightarrow{\psi} S^1 \times [-1,1]$$

where $\psi(x,y,z) = (u+iv,\ w)$



Exercise. Verify that φ, ψ are homeomorphisms.

Remark. Möbius strip has every nbhd of the form of a product $U \times V$ for $U \subset S^1$ and $V \subset [a,b]$.

But the whole space is not $S^1 \times [a,b]$.